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A SHARP BOUND FOR THE NUMBER OF SETS THAT PAIRWISE INTERSECT AT k POSITIVE VALUES

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In this paper we prove that if $\mathcal{L}$ is a set of k positive integers and $\{A_1,\dots,A_m\}$ is a family of subsets of an n-element set satisfying $|A_i\cap A_j|\in\mathcal{L}$, for all $1\le i<j\le m$, then $m\le\sum_{i=0}^{k}\binom{n-1}{i}$. The case $k=1$ was proven 50 years ago by Majumdar.

1. Introduction

Throughout this paper X will denote the set $[n]=\{1,2,\dots,n\}$ and $\mathcal{L}=\{l_1,l_2,\dots,l_k\}$ will denote a set of k arbitrary positive integers. Let $\mathcal{A}=\{A_1,A_2,\dots,A_m\}$ denote a collection of subsets of X such that $|A_i\cap A_j|\in\mathcal{L}$ for all $1\le i<j\le m$. We will sometimes refer to $\mathcal{A}$ as an *$\mathcal{L}$-intersecting family*. In this paper we are concerned with finding upper bounds for $|\mathcal{A}|$. In particular, by modifying the results in [8] and [9], we prove that an $\mathcal{L}$-intersecting family of subsets of X has at most $\sum_{i=0}^{k}\binom{n-1}{i}$ sets. The case $k=1$ was proven 50 years ago by Majumdar [5].

2. Main Result

First we must digress and discuss notation.

Notation. Let $\binom{X}{k}$ denote all the k-element subsets of X. Let $X^*=\{x_1,x_2,\dots,x_n\}$ be a collection of n variables and let $\binom{X^*}{k}$ (for $k\ge1$) denote the set of all k-term multilinear monomials from X^* (e.g. $x_1x_2\cdots x_k\in\binom{X^*}{k}$).

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Let

$$\sum \binom{X^*}{k} = \sum_{x_{i_1}x_{i_2}\cdots x_{i_k} \in \binom{X^*}{k}} x_{i_1}x_{i_2}\cdots x_{i_k}$$

and let $\binom{X^*}{0}=1$. Given $\mathcal{L}=\{l_1,l_2,\ldots,l_k\}$ define

$$g_{\mathcal{L}}(y) = \prod_{1\le i\le k}(y-l_i).$$

Since $g_{\mathcal{L}}(y)$ is a monic polynomial in y of degree k, we can write it (by a change of basis) in the form $g_{\mathcal{L}}(y)=\sum_{h=0}^{k} c_h\binom{y}{h}$, where $c_0,c_1,\ldots,c_k$ are real numbers independent of y, which we will call the coefficients of $\mathcal{L}$.

Let

$$g^*_{\mathcal{L}}(x) = c_k\sum\binom{X^*}{k} + c_{k-1}\sum\binom{X^*}{k-1} + \cdots + c_0$$

where the coefficients c_i are the coefficients of $\mathcal{L}$ and $x=(x_1,x_2,\ldots,x_n)$. With each set $A_i\in\mathcal{A}$, we associate its characteristic vector $v_i=(v_{i_1},v_{i_2},\ldots,v_{i_n})\in\mathbb{R}^n$, where $v_{i_j}=1$ if $j\in A_i$ and $v_{i_j}=0$ otherwise. Note that $g^*_{\mathcal{L}}(v_i)=g_{\mathcal{L}}(|A_i|)$.

For each $A_i=\{i_1,i_2,\ldots,i_t\}$ a member of $\mathcal{A}$, let $A^*_i=\{x_{i_1},x_{i_2},\ldots,x_{i_t}\}$ be a collection of $|A_i|$ variables where $x_{i_j}\in A^*_i$ if and only if $i_j\in A_i$. Let $\binom{A^*_i}{k}$ (for $k\ge 1$) denote the set of all k-term multilinear monomials from A^*_i. Let $\binom{A^*_i}{0}=1$.

Using the same coefficients as in $g^*_{\mathcal{L}}(x)$ define $g^*_{A_i}(x) = c_k\sum\binom{A^*_i}{k} + c_{k-1}\sum\binom{A^*_i}{k-1}+\ldots+c_0$ where $x=(x_1,x_2,\ldots,x_n)$. Note that $g^*_{A_i}(v_i)=g^*_{\mathcal{L}}(v_i)=g_{\mathcal{L}}(|A_i|)$ and that $g^*_{A_i}(v_j)=g_{\mathcal{L}}(|A_i\cap A_j|)=0$ (for all $j\ne i$).

Claim 1. *The coefficients of $\mathcal{L}$ alternate in sign.*

Proof of Claim 1. The proof is an induction on k. The basis case $k=1$ is trivial. Let $\hat{\mathcal{L}}=\{l_1,l_2,\ldots,l_{k-1}\}$ and assume that our claim is true for $g_{\hat{\mathcal{L}}}(y)$. Thus $g_{\hat{\mathcal{L}}}(y)=c_{k-1}\binom{y}{k-1}+c_{k-2}\binom{y}{k-2}+\cdots+c_0\binom{y}{0}$ with $c_{k-2},c_{k-4},\ldots$ all negative and $c_{k-1},c_{k-3},\ldots$ all positive. Now $g_{\mathcal{L}}(y)=g_{\hat{\mathcal{L}}}(y)(y-\ell_k)$ and note that $\ell_k\ge k$. Thus

$$\begin{aligned} g_{\mathcal{L}}(y) = {} & c_{k-1}\frac{k}{k}\binom{y}{k-1}(y-(k-1)+k-1) + c_{k-1}\binom{y}{k-1}(-\ell_k) + \\ & + c_{k-2}\frac{k-1}{k-1}\binom{y}{k-2}(y-(k-2)+k-2) + c_{k-2}\binom{y}{k-2}(-\ell_k) + \\ & + \cdots + c_0\binom{y}{0}y + c_0\binom{y}{0}(-\ell_k). \end{aligned}$$

So

$$g_{\mathcal{L}}(y) = c_{k-1} \cdot k\binom{y}{k} + c_{k-1} \cdot ((k-1) - \ell_k)\binom{y}{k-1} + \\ + c_{k-2} \cdot (k-1)\binom{y}{k-1} + c_{k-2}((k-2) - \ell_k)\binom{y}{k-2} + \\ + c_{k-3}(k-2)\binom{y}{k-2} + c_{k-3}((k-3) - \ell_k)\binom{y}{k-3} + \cdots + c_0(-\ell_k)\binom{y}{0}$$

where the last term has the opposite sign of c_0. ∎

From now on we will view the multilinear polynomials $g^*_{A_i}(x)$ as polynomials from $\mathbb{R}^n$ to $\mathbb{R}$.

Claim 2. *The $g^*_{A_i}(x)$'s are linearly independent.*

Proof of Claim 2. Suppose our claim is false, then there must exist some linear combination of the $g^*_{A_i}(x)$'s equal to the zero polynomial, say:

$$\alpha_1 g^*_{A_1}(x) + \alpha_2 g^*_{A_2}(x) + \cdots + \alpha_m g^*_{A_m}(x) = 0 \tag{1}$$

For the moment treat the α_i 's as variables and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. For $Y \subset X$ let $L_Y(\alpha) = \sum_{j: Y \subset A_j} \alpha_j$ and consider the linear system

$$L_Y(\alpha) = 0 \text{ for all } Y \subseteq X \text{ with } 0 \le |Y| \le k \tag{S_1}$$

In [9] it was shown that if $\mathcal{A}$ is an $\mathcal{L}$-intersecting family of subsets of X, then S_1 has only the trivial solution. Now let Y_0 be an arbitrary subset in X with $|Y_0| \le k$ and $Y_0 \subset A_i$ for some i. Without loss of generality assume that $Y_0 = \{1, 2, \ldots, r\}$ and consider the monomial $x_1 x_2 \cdots x_r$. Make the following substitution to the LHS of (1) for $x = (x_1, x_2, \cdots, x_n)$ let $x_i = y$ if $i \in Y_0$ and $x_i = 0$ otherwise. The LHS of (1) becomes a polynomial in y of degree r with the coefficient of y^r equal to $c_r\ L_{Y_0}(\alpha)$. Since this polynomial is equivalent to the zero polynomial for all possible values of y, we see that this implies that $L_{Y_0}(\alpha) = 0$. (Here we are using the fact that the coefficients of $\mathcal{L}$ are nonzero and the same for all of the $g^*_{A_i}$'s.) Since this must hold true for all $Y \subseteq X$ with $0 \le |Y| \le k$ we see that α must be a solution to S_1. Hence $\alpha = (0, 0, \ldots, 0)$ and the $g^*_{A_i}$'s are linearly independent. ∎

Let $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_k\}$ with $1 \le \ell_1 < \ell_2 < \ldots < \ell_k$ and let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an $\mathcal{L}$-intersecting family with $|A_i| \le |A_j|$ whenever $i < j$ and assume that $|A_1| > \ell_1$. All polynomials in n-variables discussed below will be considered as multilinear polynomials where each occurence of x_i^p $(p > 1)$ has been replaced by x_i. Let

$$f_{A_i}(x) = \prod_{\ell_j < |A_i|} (v_i \bullet x - \ell_j)$$

where $v_i \bullet x$ is the standard inner product in $\mathbb{R}^n (x = (x_1, \ldots, x_n))$ and $v_i = (v_{i_1}, \ldots, v_{i_n})$ is the characteristic vector of A_i.

Note that $f_{A_i}(v_i) \neq 0$ and that $f_{A_i}(v_j) = 0$ whenever $j < i$ (since $0^2 = 0$ and $1^2 = 1$). It is easy to show that the f_{A_i}'s are linearly independent.

We are now ready to prove our main result.

Theorem 1. *If $\mathcal{L} = \{\ell_1, \ell_2, \ldots, \ell_k\}$ is a set of k positive integers and $\mathcal{A} = \{A_1, \ldots, A_m\}$ is a family of subsets of an n-element set satisfying $|A_i \cap A_j| \in \mathcal{L}$, for all $1 \leq i < j \leq m$, then $|\mathcal{A}| \leq \sum_{i=0}^{k} \binom{n-1}{i}$.*

Proof. Let $\ell_1 < \ell_2 < \ldots < \ell_k$. Without loss of generality if $|A_i| = \ell_1$ for some i then we may assume that $n \notin A_i$.

We may also assume (after relabeling) that for $1 \leq i \leq r, n \notin A_i$ and that for $i > r, n \in A_i$. Furthermore, we will assume that if $r < i < j$ then $|A_i| \leq |A_j|$. For $i = 1, \ldots, m$ let us define a polynomial $p_i(x)$ in n variables as follows:

$$p_i(x) = g^*_{A_i}(x) \text{ for } 1 \leq i \leq r \text{ and } p_i(x) = f_{A_i}(x) \text{ for } i > r.$$

From now on we will view all multilinear polynimials as polynomials from $\mathbb{R}^n$ to $\mathbb{R}$. Let $P_{\leq r} = \{p_i(x) : 1 \leq i \leq r\}$ and $P_{>r} = \{p_i(x) : i > r\}$.

We have already shown that $P_{\leq r}$ is a collection of linearly independent polynomials and that $P_{>r}$ is also a collection of linearly independent polynomials. We claim that $P_{\leq r} \cup P_{>r}$ is a collection of linearly independent polynomials. Suppose not. Then there must exist some linear combination of the p_i's that equals the zero polynomial, say:

$$(2) \qquad \alpha_1 p_1(x) + \ldots + \alpha_r p_r(x) + \alpha_{r+1} p_{r+1}(x) + \ldots + \alpha_m p_m(x) = 0$$

with some $\alpha_i \neq 0$ for some $i \leq r$ and with some $\alpha_j \neq 0$ for some $j > r$. Let j_0 be the least subscript greater than r such that $\alpha_{j_o} \neq 0$. Substituting the characteristic vector v_{j_0} into the LHS of (2) we get $\alpha_{j_0} p_{j_0}(v_{j_0}) = \alpha_{j_0} f_{A_{j_0}}(v_{j_0}) = 0$ which is impossible. Therefore the p_i's are linearly independent.

Label the sets in $\binom{X - \{n\}}{0} \cup \binom{X - \{n\}}{1} \cup \ldots \cup \binom{X - \{n\}}{k-1}$ with the labels B_i for $i = 1, \ldots, q = \sum_{i=0}^{k-1} \binom{n-1}{i}$ such that $|B_i| \leq |B_j|$ when $i < j$. Let w_i be the characteristic vector of B_i and let

$$h_{B_i}(x) = \prod_{j \in B_i} x_j$$

for $i>1$. Note that the w_i are in $\Omega=\{0,1\}^n$.

For $i=1,\dots,q$ let us define a multilinear polynomial g_{B_i} in n variables as follows:

$$g_{B_1}(x) = x_n - 1$$

and

$$g_{B_i}(x) = x_n h_{B_i}(x) - h_{B_i}(x) \quad \text{for } i > 1.$$

Clearly, the g_{B_i}'s are linearly independent (use the fact that $g_{B_i}(w_i)\neq 0$).

Next we claim that the polynomials in $P_{\leq r}$ together with the g_{B_i}'s is a collection of linearly independent polynomials. Suppose not. Then there must exist some linear combination of the p_i's ($i \leq r$) and the g_{B_j}'s that equals the zero polynomial, say:

$$\alpha_1 p_1(x) + \ldots + \alpha_r p_r(x) + \beta_1 g_{B_1}(x) + \ldots + \beta_q g_{B_q}(x) = 0 \tag{3}$$

with some $\alpha_i\neq 0$ and some $\beta_j\neq 0$.

Suppose $\beta_{j_0}\neq 0$ and assume that $|B_{j_0}|=t$. Make the following substitution to the LHS of (3) for $x=(x_1,\dots,x_n)$ let $x_i=y$ if $i\in B_{j_0}\cup\{n\}$ and $x_i=0$ otherwise. Then the LHS of (3) becomes a polynomial in y of degree $t+1$ with the coefficient of y^{t+1} equal to β_{j_0}. Since this polynomial is equivalent to the zero polynomial for all possible values of y, we see that $\beta_{j_0}=0$ a contradiction. Therefore the polynomials p_i's ($i\leq r$) together with the g_{B_j}'s are linearly independent. It is also easy to see that the p_i's (with $i>r$) together with the g_{B_j}'s is a collection of linearly independent polynomials (all characteristic vectors $v_i=(v_{i_1},\dots,v_{i_n})(i>r)$ have $v_{i_n}=1$).

Finally, we claim that the p_i's together with the g_{B_j}'s are linearly independent. Suppose not. Then there must exist some linear combination of the p_i's and g_{B_j}'s that equals the zero polynomial, say:

$$\begin{aligned} \alpha_1 p_1(x) + \ldots + \alpha_r p_r(x) + \alpha_{r+1} p_{r+1}\ (x) + \ldots + \alpha_m p_m(x) + \beta_1 g_{B_1}(x) + \\ + \ldots + \beta_q g_{B_q}(x) = 0 \end{aligned} \tag{4}$$

with some $\alpha_i\neq 0$ for $i\leq r$, some $\alpha_j\neq 0$ for $j>r$, and some $\beta_k\neq 0$.

Let j_0 be the least subscript greater than r such that $\alpha_{j_0}\neq 0$. Substituting the characteristic vector v_{j_0} for x into equation (4) we have, $\alpha_{j_0}p_{j_0}(v_{j_0}) = \alpha_{j_0} f_{A_{j_0}}(v_{j_0})=0$ and this is an contradiction. Therefore our p_i's and g_{B_j}'s are linearly independent. Now each g_{B_j} and p_i can be written as a linear combination of the multilinear monomials of degree $\leq k$. The number of such monomials is $\sum_{i=0}^{k}\binom{n}{i}$. We have $q=\sum_{i=0}^{k-1}\binom{n-1}{i} g_{B_j}$'s, thus $m\leq\sum_{i=0}^{k}\binom{n}{i} - \sum_{i=0}^{k-1}\binom{n-1}{i}=\sum_{i=0}^{k}\binom{n-1}{i}$ and we are done. ∎

As a corollary we get the following well-known result of Frankl and Wilson [3].

Corollary (Frankl and Wilson). *Let $\mathcal{L}^*$ be a collection of k non-negative integers and let $\mathcal{A}=\{A_1,\dots,A_m\}$ be a collection of subsets of X. If $|A_i \cap A_j| \in \mathcal{L}^*$ for all $A_i, A_j, \in \mathcal{A}$, then $|\mathcal{A}| \le \sum_{i=0}^{k}\binom{n}{i}$.*

Proof. Suppose there exists a set $\mathcal{L}^* = \{0 = l_1, l_2, \dots, l_k\}$ and a set $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ such that $|A_i \cap A_j| \in \mathcal{L}^*$, for all $A_i, A_j \in \mathcal{A}$ but $|\mathcal{A}| > \sum_{i=0}^{k}\binom{n}{i}$. Create a new collection of sets $\mathcal{A}' = \{A_1', A_2', \dots, A_m'\}$ with $A_i' = A_i \cup \{n+1\}$. Then $\mathcal{A}'$ is a collection of subsets of $X' = X \cup \{n+1\}$ such that$|A_i' \cap A_j'| \in \mathcal{L} = \{1, l_2+1, \dots, l_i+1\}$, for all $A_i', A_j' \in \mathcal{A}'$ and $|\mathcal{A}'| > \sum_{i=0}^{k}\binom{(n+1)-1}{i}$, which is a contradiction. ∎

3. Conjectures

Conjecture 1. Let p be a prime and let L and K be disjoint subsets of $\{0, 1, 2, \dots, p-1\}$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ be a collection of subsets of $X = \{1, 2, \dots, n\}$ with the property that $|A_i \cap A_j| \bmod p \in L$ (for $i \ne j$) and $|A_i| \bmod p \in K$. Then we claim $|\mathcal{A}| \le \binom{n}{|L|}$ and this is best possible. (So far we have $|\mathcal{A}| < \binom{n}{|L|} + \binom{n}{|L|-1}$ for n sufficiently large [8].)

The next conjecture can be considered as a Bollobas Type Theorem related to perfect graphs.

Conjecture 2. Let $\mathcal{A} = \{A_1, A_2, \dots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be two collections of subsets of an n-element set. Let $\mathcal{L} = \{l_1, l_2, \dots, l_k\}$ be a collection of k positive integers. Assume that for $i \ne j$ we have $|A_i \cap B_j| \in \mathcal{L}$ and that $|A_i \cap B_i| = 0$, then we conjecture that $m \le \binom{n}{k}$.

This bound is sharp – just take all k-element subsets and all $n-k$-element subsets. This is related to perfect graphs via Padberg's conditions on p critical graphs. (Clique's are the A_i's and the independent sets are the B_j's and of course $k=1$).

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